Lectures 2-3

Wednesday, January 22, 2020 6:06 AM

We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ by $z = (z_1, \dots, z_n)$, $x = (x_1, x_2, \dots, x_{2n})$ and

$$z_j = x_{2j-1} + i x_{2j}, \quad j = 1, \dots, n.$$

Introduce $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2j-1}} - i \frac{\partial}{\partial x_{2j}}\right)$, $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2j-1}} + i \frac{\partial}{\partial x_{2j}}\right)$ and

$\partial = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j$ (i.e. $\partial u = \sum_{j=1}^{n} \frac{\partial u}{\partial z_j} dz_j$), $\bar{\partial} = \sum_{j=1}^{n} \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j$. $\quad \sum_{k=1}^{2n} u_{x_k} dx_k$

$\Rightarrow \partial u$ is $(1,0)$ form, $\bar{\partial} u$ is $(0,1)$ form, $du = \partial u + \bar{\partial} u$ as in $\mathbb{C}^1$.

<u>Higher degree forms</u>. A (p,q) form f is a $(p+q)$-form of form

$$f = \sum_{|I|=p} \sum_{|J|=q} f_{I,J}\, dz^I \wedge d\bar{z}^J,$$ where I, J are <u>multi-indices</u> of <u>length</u> p, q.

e.g. $I = (i_1, \dots, i_p)$, $i_k \in \{1, \dots, n\}$. Also,

$$dz^I = dz_{i_1} \wedge \dots \wedge dz_{i_p}.$$

- Any m-form $g = \sum_{|M|=m} g_M\, dx^M$ can be decomposed $g = \sum_{p+q=m} g^{pq}$, where g^{pq} is a (p,q) form.

- $\partial, \bar{\partial}$ act on (p,q) forms (and linearly on m-forms) by

$$\bar{\partial} f = \sum_{\substack{|I|=p \\ |J|=q}} \bar{\partial} f_{I,J}\, dz^I \wedge d\bar{z}^J,$$

so that ∂f is $(p+1,q)$ form, $\bar{\partial} f$ is $(p,q+1)$-form and

$|J|=q$

so that ∂f is $(p+1,q)$-form, $\bar\partial f$ is $(p,q+1)$-form and

$$df = \partial f + \bar\partial f.$$

- Differential form calculus applies to $\partial, \bar\partial$. E.g. $\partial^2 = \bar\partial^2 = 0$.

<u>Major Ex</u>. u is a function (0-form), then $\bar\partial u$ is a $(0,1)$ form $\bar\partial u = \sum_{j=1}^{n} u_{\bar z_j}\, d\bar z_j$. <u>Note</u>. A $(0,1)$-form has n components, which means that the <u>$\bar\partial$-equation</u> $\bar\partial u = f$ is <u>overdetermined system</u> when $n>1$ and single (always solvable) equation when $n=1$.

<u>Rem</u>. • The $\bar\partial$-equation is not very interesting when $n=1$ but <u>fundamental</u> when $n>1$.

- When $n>1$, f a $(0,1)$-form, then a necessary condition for $\bar\partial u = f$ to be solvable is $\bar\partial f = 0$. We will come back to $\bar\partial$-equation after more prelims.

<u>Holomorphic functions</u>.

<u>Def</u> 0. Let $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}$ be C^1. Then f is <u>holomorphic</u> if

$$\bar\partial f = 0 \iff df \text{ is } (1,0)\text{-form}.$$

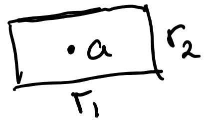
$$\bar{\partial} f = 0 \iff df \text{ is } (1,0)\text{-form.}$$

Study first in simple geometry.

<u>Def 1.</u> A <u>polydisk</u> (open polydisk) $\Omega = D^n \subseteq \mathbb{C}^n$ is a domain of the form

$D_1 \times D_2 \times \dots \times D_n$, where $D_j = \{ z \in \mathbb{C} : |z - a_j| < r_j \}$.

$a = (a_1, \dots, a_n)$ is <u>center</u>, $r = (r_1, \dots, r_n)$ is poly-<u>radius</u>.

(Like rectangle ).

The boundary ∂D^n is not smooth, but has smooth stratification.

<u>Ex.</u> The bidisk $\mathbb{D}^2 \subseteq \mathbb{C}^2$, $\mathbb{D} = \{ |z| < 1 \}$. $\partial \mathbb{D}^2 = B_0 \cup B_1 \cup B_2$,

$B_0 = \{ (z,w) : |z| = |w| = 1 \}$, $B_1 = \{ (z,w) : |z| = 1, |w| < 1 \}$, $B_2 = \dots$.

2-torus $S^1 \times S^1$ 3-dim, foliated by disks.

<u>Def 2.</u> The boundary component $\partial_0 D^n = \{ z \in \mathbb{C}^n : |z_1| = \dots = |z_n| = 1 \}$ is called <u>distinguished boundary</u>.

Note that $\partial_0 D^n = \partial D_1 \times \dots \times \partial D_n$ is n-torus.

<u>Cauchy's Formula (in polydisks)</u>. Let D^n open polydisk, $u : \overline{D^n} \to \mathbb{C}$ continuous that is <u>separately holomorphic</u> in D^n, i.e.

$\forall z \in D^n$, $z \to u(z_1, \dots, \overset{j\text{-th}}{z}, \dots, z_n)$ is holomorphic in D_j.

$\forall z \in D^n$, $\zeta \to u(z_1, \dots, \overset{j\text{:th}}{\zeta}, \dots, z_n)$ is holomorphic in D_j.

Then
$$u(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D^n} \frac{u(\zeta)}{(\zeta_1 - z_1)\dots(\zeta_n - z_n)} \, d\zeta_1 \dots d\zeta_n$$

Pf. Let's do $D^2 \subseteq \mathbb{C}^2$. Fix $z = (z_1, z_2) \in D^2$.

① Since $\zeta \to (\zeta, z_2)$ is holomorphic in D_1, cont. up to ∂D_1, 1-dim Cauchy $\Rightarrow$
$$u(z_1, z_2) = \frac{1}{2\pi} \int_{\partial D_1} \frac{u(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_1. \quad (1)$$

② We know $w \to u(\zeta_1, w)$ is holom. in D_2, $\forall \zeta_1 \in D_1$. What about $\zeta_1 \in \partial D_1$? Well, pick $\zeta_1 \in \partial D_1$, take $\xi_n \in D_1$, $\xi_n \to \zeta_1$, $u_n(w) = u(\xi_n, w)$. Each u_n is holom. in D_2, cont. on $\overline{D_2}$, and $u_n \to u(\zeta_1, \cdot)$ uniformly in $\overline{D_2}$ by uniform cont. of u on $\overline{D^2}$. (in particular, uniform convergence on compacts).

$\Rightarrow$ the limit is holom. in D_2 and cont. on $\overline{D_2}$.

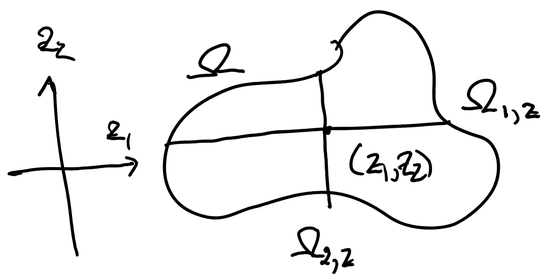
Applying 1-dim Cauchy to integrand in (1) $\Rightarrow$
$$u(z) = \frac{1}{2\pi i} \int \left(\frac{1}{2\pi i} \int \frac{u(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \right) \frac{d\zeta_1}{\zeta - z_2},$$

$$u(z) = \frac{1}{2\pi i}\int_{\partial D_1}\left(\frac{1}{2\pi i}\int_{\partial D_2}\frac{u(\zeta_1,\zeta_2)}{\zeta_2 - z_2}\,d\zeta_2\right)\frac{d\zeta_1}{\zeta - z_2},$$

which $\Rightarrow$ ▨

<u>Cor 1.</u> Let $\Omega \subseteq \mathbb{C}^n$ domain, and u <u>cont.</u> in Ω and <u>sep. hol.</u> in Ω; i.e. $\forall z \in \Omega$, $\zeta \to u(z_1, \ldots, \overset{j\text{-th}}{\zeta}, \ldots z_n)$ is hol. for $|\zeta - z_j| < \varepsilon$ (ε so small that ↑ stays in Ω).
Then, $u \in C^\infty$ and holomorphic in Ω.

<u>Rem.</u> Fix $z \in \Omega$ and let $\Omega_{j,z} = \{\zeta \in \mathbb{C} : (z_1, \ldots, \zeta, \ldots, z_n) \in \Omega\}$, "j-slice at z".



Since $\zeta \to u(z_1, \ldots, \zeta, \ldots, z_n)$ is defined in $\Omega_{j,z}$ and locally holom. at each point (by assumption), we see that in fact it is holom. in all of $\Omega_{j,z}$. Thus, local notion of sep. hol. is the same as the one considered in CF.

<u>Pf of Cor 1.</u> Conclusion is local. For any $a \in \Omega$ $\exists$ polydisk D^n of suff. small polyradius $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ centered at a s.t. $D^n \subseteq \Omega$. In D^n, we can represent $u(z)$ by Cauchy's Thm.
The integrand, as fcn of $z \in D^n$ for fixed $\zeta \in \partial_0 D^n$, is holo.
Since we may move $\bar\partial$ inside integral ($\partial_0 D^n$ compact and $\bar\partial_z$ on integrand uniformly bdd), we conclude u is holo.
Similarly, since integrand is C^∞ in same sense, we

$\partial_{\bar z}$ on integrand uniformly bdd), we conclude u is holo.
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<u>Rem.</u> Many properties that follow from Cauchy's Formula in $\mathbb{C}$ follow in same way in $\mathbb{C}^n$ from CF in D^n, e.g., <u>maximum principle</u>, <u>unif. conv. on compacts $\Rightarrow$ holo.</u>, etc. We will mention these as we need them. Important ones follow:

<u>Power Series Expansion.</u>

<u>Def. 3.</u> A series $\sum_\alpha a_\alpha(z)$ converges <u>normally</u> in $\Omega \subseteq \mathbb{C}^n$ if

countable sum

$$\sum_\alpha \sup_K |a_\alpha(z)| \text{ converges}, \ \forall \text{ compacts } K \subset\subset \Omega.$$

<u>Rem.</u> Normal conv. $\Rightarrow \sum_\alpha a_\alpha(z)$ converges to function $a(z)$. If a_α are cont., then a is cont. If a_α are holo., then a is holo.

<u>Thm 1.</u> If u is holo. in polydisk $D^n \subseteq \mathbb{C}^n$ centered at $a \in \mathbb{C}^n$, then
$$u(z) = \sum_{\alpha \in \mathbb{Z}_+^n} u_\alpha (z-a)^\alpha \ ; \quad u_\alpha = \frac{1}{\alpha!} \frac{\partial^\alpha u}{\partial z^\alpha}(a)$$
Abbr. $\partial^\alpha u$ or u_{z^α}

with normal convergence in D^n.

<u>Note:</u> Here, $(z-a)^\alpha = (z_1-a_1)^{\alpha_1} \cdots (z_n-a_n)^{\alpha_n}$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.

Pf. Follows from Cauchy's Formula as in 1 variable. ▨

Cauchy's Estimates. Let $D^n \subseteq \mathbb{C}^n$ be polydisk of polyradius r, centered at a. If u is holo. in D^n, and $|u| \le M$ in D^n, then

$$|u_{z^\alpha}(a)| \le \frac{M\alpha!}{r^\alpha}.$$

Pf. Follows from CF as in 1 variable. ▨

Another consequence of the power series expansion (as in $\mathbb{C}$) is unique continuation: If u is holom., $u_{z^\alpha}(a) = 0$ $\forall \alpha \in \mathbb{Z}_+^n$, then $u \equiv 0$ in an open polydisk D^n centered at a. If u holom. in $\Omega \subseteq \mathbb{C}^n$, Ω connected, and $u \equiv 0$ on open polydisk $D^n \subseteq \Omega$, then $u \equiv 0$ in Ω.